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Models of traps and glass phenomenology

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Abstract. We study various models of independent particles hopping between energy 'traps' with a density of energy barriers $\rho(E)$, on a d-dimensional lattice or on a fully connected lattice. If $\rho(E)$ decays exponentially, a true dynamical phase transition between a high-temperature 'liquid' phase and a low-temperature 'aging' phase occurs. More generally, however, one expects that for a large class of $\rho(E)$, 'interrupted' aging effects appear at low enough temperatures, with an ergodic time growing faster than exponentially. The relaxation functions exhibit a characteristic shoulder, which can be fitted as stretched exponentials. A simple way of introducing interactions between the particles leads to a modified model with an effective diffusion constant in energy space, which we discuss in detail.

1. Introduction

Many very different glass formers exhibit surprisingly similar properties. For example, a common experimental feature is the 'shouldering' of the relaxation laws as the temperature is decreased [1, 2]. More precisely, the relaxation of the density fluctuations evolves from a simple Debye exponential at high temperatures (liquid) to a two-step process at lower temperature, where the correlation function first decays rather quickly to a 'plateau' (β relaxation), and then departs from this plateau value on a much longer time scale $\tau(T)$. This relaxation time $\tau(T)$ grows extremely fast as the temperature is decreased, in any case faster than $\exp(\Delta/T)$. A very successful description of this divergence is the Vogel–Fulcher law: $\Gamma_0\tau(T) \sim \mathrm{e}^{\Delta/(T-T_0)}$, where Γ_0^{-1} is a microscopic time scale [3]. However, other functional forms, such as $\Gamma_0\tau(T) \sim \mathrm{e}^{(\Delta/T)^2}$, give reasonable fits of the data [4, 5]. When $\tau(T)$ becomes of the order of the typical experimental time scales (say, a day), the system is conventionally called a 'glass', which is thus inherently out of equilibrium.

A remarkable and popular theory of dynamical processes in supercooled liquids is the so-called mode-coupling theory (MCT), developed by Götze and others [1]. Starting from a family of schematic equations which include a nonlinear, retarded feedback of the density fluctuations, one can show that there exists an 'ideal glass' transition temperature T_c, below which the correlation function does not decay to zero ('broken ergodicity'). A notable prediction of the theory is the existence of the two regimes β and α mentioned above, and a power-law divergence of the 'slow' time scale $\tau(T)$ as $(T - T_c)^{-\gamma}$. However, a detailed comparison with the experiments [6] shows that the transition temperature T_c, if it exists,

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is much higher than the Vogel–Fulcher temperature, leaving a whole temperature interval $[T_0, T_c]$ where MCT predicts a partial freezing (i.e. the α regime disappears) while the experimental relaxation time is still finite (and behaves *à la* Vogel–Fulcher): the evidence for a (smeared) critical temperature is thus not very compelling. A way to circumvent this difficulty could be, as recently proposed in [9], to work deep below T_c, where an extension of the MCT could be unambiguously tested.

However, the physical status of the MCT is not yet very clear: the MCT equations are formally identical to those describing some mean field, infinite-dimensional models of spin-glasses [7–9], where the presence of *quenched* disorder is assumed from the start. Further work should clarify the interpretation of these underlying 'spin' degrees of freedom, the importance of finite dimensionality effects (in particular to allow the existence of activated processes, which are presumably absent in infinite dimension) and the justification of introducing quenched disorder by hand, rather than letting it be 'self-induced' by the dynamics [10].

In view of these difficulties, it is interesting to investigate in detail alternative phenomenological descriptions. Among quite a few other approaches [1, 11], a particularly simple picture, advocated by many authors over the years, is the following: each particle is in a 'cage', i.e. a potential well created by its neighbours, from which it can escape through thermal activation [4, 5, 12–15]. As the temperature is decreased, the probability distribution of local trapping times becomes very broad, and this naturally leads to a two-step relaxation, stretched exponential decays and Cole–Cole susceptibility spectra. However, two important facets of these 'trap' models have not been addressed previously in the context of glasses.

- At low enough temperatures, the relaxation times exceed the experimental time scales, and aging effects become important [17–19]. These aging effects have been recently discussed within the framework of the MCT [9, 16]. Special attention will be devoted to situations where aging is 'interrupted' [17, 20] beyond a finite, but very long, time scale—which is often the case experimentally.
- Since *a priori* all particles can move, the (random) potential well trapping any one of them is in fact not quenched but time dependent, further enhancing the probability of moving. In order to understand the glass transition, one must describe how, in a self-consistent way, all motion ceases. This problem is similar to the one alluded to above in the context of MCT.

It is the aim of the present paper to discuss three models in detail (a preliminary account of this work can be found in [15]), as well as to investigate systematically other aspects of these trap models, such as finite time properties. We also compare the results of this approach to the observed properties of glasses, and emphasize its strengths and weaknesses.

2. Mean-field model of non-interacting traps

2.1. Definition of the model

We first consider the thermal dynamics of *independent* particles in a space of traps characterized by a given probability distribution $\rho(E)$ for the depth $E \in [0, +\infty[$ of traps [17]. At temperature $T \equiv \beta^{-1}$, each particle may escape from its trap of depth E with rate $\Gamma_0 \, e^{-\beta E}$ per unit time. In the simplest 'mean-field' version, the particle chooses a new trap of depth E' with probability $\rho(E')$, but with no reference to any *spatial* structure: this will be considered below. The probability density $P(E, t)$ to be in a trap of depth E at time t

therefore evolves in time according to the following equation

$$\frac{1}{\Gamma_0}\frac{\partial P(E,t)}{\partial t} = -\mathrm{e}^{-\beta E}\ P(E,t) + \omega(t)\rho(E) \tag{1}$$

where $\omega(t) = \int_0^\infty \mathrm{d}E\, \mathrm{e}^{-\beta E} P(E,t)$ is the average hopping rate at time t. This equation has to be supplemented by some initial condition $P(E, t=0) = P_0(E)$.

A natural definition of the (two-time) correlation function within such a model is the following,

$$C(t_w + t, t_w) = \Pi(t_w + t, t_w) + q_0[1 - \Pi(t_w + t, t_w)] \tag{2}$$

where $\Pi(t_w + t, t_w)$ is the probability that the particle has not changed trap between t_w and $t_w + t$:

$$\Pi(t_w + t, t_w) = \int_0^\infty \mathrm{d}E\, P(E, t_w) \exp[-\left(\Gamma_0\, \mathrm{e}^{-\beta E}\right) t]. \tag{3}$$

q_0 is a certain number measuring the correlation between traps, which we shall set to zero for simplicity. Equation (2) assumes that no dynamics take place 'inside a trap' since $C \equiv 1$ until the particle has jumped out of the trap. If a stationary distribution can be reached at long times, the correlation function becomes *time translation invariant* and reads

$$\lim_{t_w \to \infty} C(t_w + t, t_w) = C_{\mathrm{eq}}(t) = \int_0^\infty \mathrm{d}E\, P_{\mathrm{eq}}(E) \exp[-(\Gamma_0\, \mathrm{e}^{-\beta E})t]. \tag{4}$$

Note that the above correlation function is a 'one-particle' (tracer) correlation function rather than a 'collective' density–density correlation function. It would be interesting to extend the following calculations to the latter case.

2.2. Existence of a stationary distribution

There exists a *normalizable* stationary distribution $P_{\mathrm{eq}}(E)$ at temperature $T = \beta^{-1}$ only if

$$\omega_{\mathrm{eq}}(\beta) \equiv \frac{1}{\displaystyle\int_0^{+\infty} \mathrm{d}E\, \mathrm{e}^{+\beta E}\rho(E)} > 0 \tag{5}$$

in which case

$$P_{\mathrm{eq}}(E) = \omega_{\mathrm{eq}}(\beta)\, \mathrm{e}^{+\beta E}\rho(E). \tag{6}$$

In particular, $P_{\mathrm{eq}}(E)$ always exists at infinite temperature ($\beta = 0$) and is equal to the 'bare' distribution $\rho(E)$. The transition between the high-temperature phase ($T > T_0$) where $P_{\mathrm{eq}}(E)$ exists, and the low-temperature phase $T < T_0$ where $P_{\mathrm{eq}}(E)$ ceases to be normalizable, takes place at the temperature T_0 defined by

$$\frac{1}{T_0} = \beta_0 \equiv \lim_{E\to\infty}\left[-\frac{\log\rho(E)}{E}\right]. \tag{7}$$

We may therefore distinguish the three cases:

- $T_0 = 0$ if $\rho(E)$ decays faster than exponentially at large E;
- $T_0 = \infty$ if $\rho(E)$ decays slower than exponentially at large E;
- T_0 is finite in the interesting case where $\rho(E)$ decays exponentially as $\mathrm{e}^{-\beta_0 E}$ at large E. This exponential form is suggested by the mean-field replica theory of spin-glasses [21], the random energy model [22] and by phenomenological arguments in the context of glasses [14]. It has also been found exactly in the one-dimensional 'random force' model [23] and was used to interpret anomalies in the transport properties of amorphous conductors (see

e.g. [24]). Its ubiquity in random systems is probably related to the fact that it is a stable extreme value distribution [25].

For this exponential density of states, the trapping time $\tau \equiv \Gamma_0^{-1} \mathrm{e}^{\beta E}$ is distributed as a power law for large τ:

$$\Psi(\tau) \underset{\tau \to \infty}{\propto} \frac{\Gamma_0}{(\Gamma_0 \tau)^{1+x}} \qquad x \equiv \frac{T}{T_0}. \tag{8}$$

Since this model exhibits a true finite temperature 'glass' transition, its study is most interesting. However, the case of a Gaussian density of states, where strictly speaking $T_0 = 0$, exhibits several features very similar to those observed in real glasses, and will thus also be studied in detail.

2.3. Relaxation towards the equilibrium distribution for $T > T_0$

At infinite temperature ($\beta = 0$), the solution of (1) may be directly written since $\omega(t) \equiv 1$

$$P(E,t) = \rho(E) + [P(E,0) - \rho(E)]\,\mathrm{e}^{-\Gamma_0 t}. \tag{9}$$

At finite temperature, we may analyse the solution of (1) through a Laplace transform: introducing $\hat{P}(E,\lambda) \equiv \int_0^\infty \mathrm{d}t [P(E,t) - P_{\mathrm{eq}}(E)]\,\mathrm{e}^{-\lambda \Gamma_0 t}$, one obtains

$$\hat{P}(E,\lambda) = \frac{(1/\Gamma_0)[P(E,t=0) - P_{\mathrm{eq}}(E)] + \hat{\omega}(\lambda)\rho(E)}{\lambda + \mathrm{e}^{-\beta E}} \tag{10}$$

with the definition

$$\hat{\omega}(\lambda) \equiv \int_0^\infty \mathrm{d}E'\,\mathrm{e}^{-\beta E'} \hat{P}(E',\lambda). \tag{11}$$

Solving for $\hat{\omega}(\lambda)$, one finds

$$\hat{\omega}(\lambda) = \frac{1}{\Gamma_0 \lambda} \left[\omega_{\mathrm{eq}} - \frac{\displaystyle\int_0^\infty \mathrm{d}E\, P(E,0)/(1+\lambda\,\mathrm{e}^{\beta E})}{\displaystyle\int_0^\infty \mathrm{d}E\, \rho(E)\,\mathrm{e}^{\beta E}/(1+\lambda\,\mathrm{e}^{\beta E})} \right]. \tag{12}$$

In the interesting case of an exponential density of states $\rho(E) \propto \mathrm{e}^{-\beta_0 E}$, the denominator has a regular expansion up to the order λ^n, where n is the integer part of $x - 1$ ($x = T/T_0$), followed by a singular term of order λ^{x-1}. Hence, if $P(E,0)$ is sharply peaked (e.g. $P(E,0) = \delta(E - E_0)$), the small λ expansion of $\hat{P}(E,\lambda)$ has a λ^{x-2} singularity, indicating that the difference $P(E,t) - P_{\mathrm{eq}}(E)$ decays for large times as

$$P(E,t) - P_{\mathrm{eq}}(E) \underset{t \to \infty}{\propto} t^{-(x-1)}. \tag{13}$$

This decay is, not surprisingly, precisely the same as that of the correlation function C_{eq} defined by equation (4) [15]:

$$C_{\mathrm{eq}}(t) = \int_0^\infty \mathrm{d}E \omega_{\mathrm{eq}}(\beta)\,\mathrm{e}^{+\beta E} \beta_0\,\mathrm{e}^{-\beta_0 E} \exp[-(\Gamma_0\,\mathrm{e}^{-\beta E})t] \underset{t \to \infty}{\simeq} \omega_{\mathrm{eq}}(\beta)\Gamma(x)(\Gamma_0 t)^{-(x-1)}. \tag{14}$$

Note that the relaxation time $\tau(T)$, defined as the time after which the correlation has decayed down to a certain value c, diverges *à la* Vogel–Fulcher, i.e.

$$\tau(T) \simeq \frac{1}{\Gamma_0} \left(\frac{c}{\omega_{\mathrm{eq}}(\beta)\Gamma(x)} \right)^{(T_0/(T-T_0))}. \tag{15}$$

Another usual definition of the 'terminal' time, through $\tau_1 = \int_0^\infty \mathrm{d}t\, C_{\mathrm{eq}}$, would, however, lead to a power-law divergence *at a higher temperature* $2T_0$

$$\tau_1 = \frac{\omega_{\mathrm{eq}}(\beta)\beta_0}{\Gamma_0}\int_0^\infty \mathrm{d}E\, \mathrm{e}^{(2\beta-\beta_0)E} \underset{T>2T_0}{=} \frac{\omega_{\mathrm{eq}}(\beta)}{\Gamma_0}\left(\frac{T}{T-2T_0}\right). \tag{16}$$

The existence of *two* characteristic temperatures in this model was recently emphasized in the context of glasses by Odagaki [14]. If $\rho(E)$ decays faster than exponentially (say as $\exp(-E^{1+\nu})$), the relaxation towards equilibrium is faster than any power law, since the small λ expansion of $\hat{P}(E,\lambda)$ is regular. However, slower than exponential relaxation (such as $\exp -(\log t)^{1+\nu}$) is expected. More generally, the long time behaviour of $P(E,t)-P_{\mathrm{eq}}(E)$ is the same as that of C_{eq} for large t.

2.4. Non-normalizability and aging in the low-temperature phase

When there is no equilibrium distribution (5), one expects that the dynamics will never become stationary [17, 26, 27, 15], and this leads to aging effects. One may look at large time for a scaling solution for equation (1). We introduce the natural dimensionless scaling variable $u = \mathrm{e}^{\beta E}/\Gamma_0 t$ and a dimensionless function ϕ normalized as $\int_0^\infty \mathrm{d}u\, \phi(u) = 1$ and try to find the asymptotic distribution function through the form

$$P(E,t) \underset{t\to\infty}{\simeq} \beta u \phi(u). \tag{17}$$

The resulting equation for ϕ is

$$u^2\frac{\mathrm{d}\phi}{\mathrm{d}u} + (u-1)\phi(u) = -\frac{1}{\beta}\rho\left(\frac{\log(\Gamma_0 t u)}{\beta}\right)\int_{1/\Gamma_0 t}^\infty \mathrm{d}v \frac{\phi(v)}{v}. \tag{18}$$

The left-hand side presents a t independent limit as $t\to\infty$ only if the integral on the left-hand side diverges in such a way that it compensates exactly the decay of $\rho(\log(\Gamma_0 t u)/\beta)$ as $t\to\infty$.

Consider for example the case of an exponential density of states $\rho(E) = \beta_0\, \mathrm{e}^{-\beta_0 E}$, for which there is no equilibrium distribution when $x \equiv T/T_0 \in]0,1[$.

In order to get some sensible limit of equation (18) as $t\to\infty$, the function ϕ must present the singularity

$$\phi(u) \underset{u\to 0}{\simeq} \gamma u^{-x} \tag{19}$$

where γ is some normalization constant. The solution of the resulting equation

$$u^2\frac{\mathrm{d}\phi}{\mathrm{d}u} + (u-1)\phi(u) = -\gamma u^{-x} \tag{20}$$

satisfying the normalization condition $\int_0^\infty \mathrm{d}u\, \phi(u) = 1$ reads

$$\phi(u) = \frac{\sin \pi x}{\pi \Gamma(x)}\frac{1}{u}\,\mathrm{e}^{-1/u}\int_0^{1/u} \mathrm{d}y\, y^{x-1}\,\mathrm{e}^{y}. \tag{21}$$

The consequence of the scaling form (17) on the correlation function C or Π defined as the probability for the particle to remain in the same trap during the interval $[t_w, t_w + t]$ (see equation (2))

$$\Pi(t_w + t, t_w) = \int_0^\infty \mathrm{d}E\, P(E,t_w)\,\mathrm{e}^{-(\Gamma_0\,\mathrm{e}^{-\beta E})t} \tag{22}$$

is the aging behaviour

$$\Pi(t_w, t_w + t) \simeq \int_0^\infty \mathrm{d}u\, \phi(u)\,\mathrm{e}^{-1/u(t/t_w)} \tag{23}$$

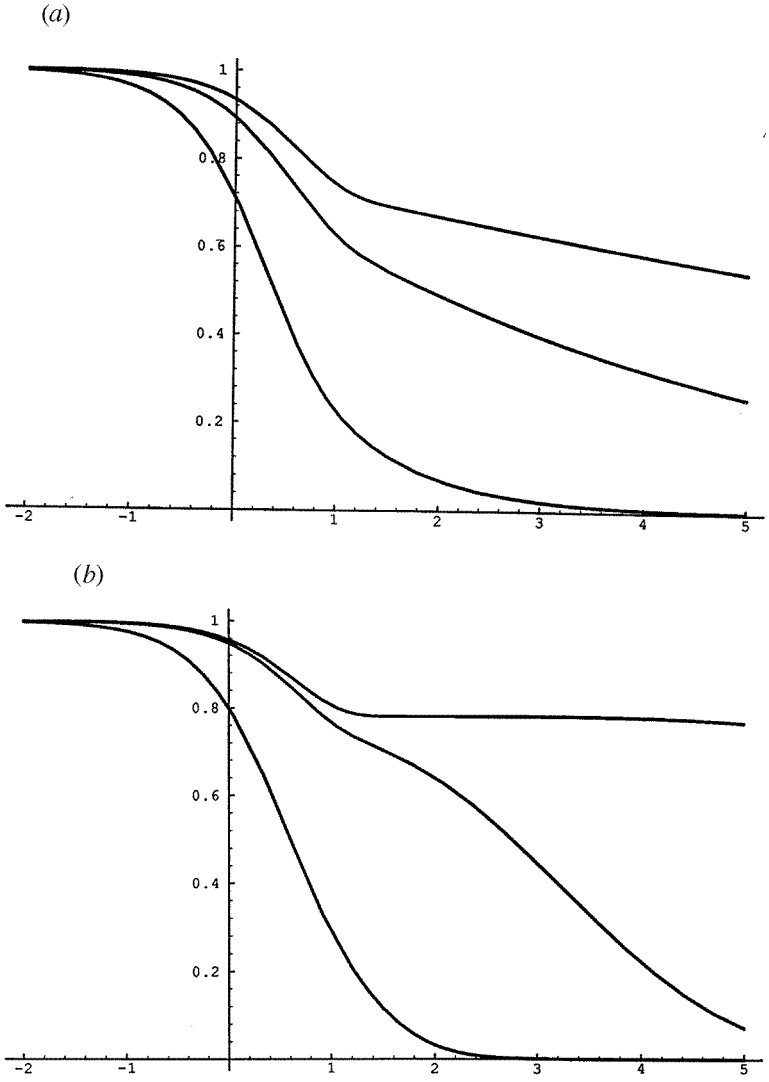

Figure 1. Correlation function $C_{\rm eq}$(t) (multiplied by a 'mock' β relaxation $C_\beta(t) \equiv \exp(-q^2 r^2(t)/2)$, where $r(t)$ describes a diffusive motion in an harmonic potential well: $r^2(t) = \xi_0^2[1 - \exp(-t/\tau_0)]$; ξ_0 can be thought as the 'size' of the cage, and ξ_0^2/τ_0 of the order of the high-temperature diffusion constant). (a) Exponential density of states. A plot of $C_{\rm eq}C_\beta(t)$ against $\log_{10}(\Gamma_0 t)$ for $q\xi_0 = 0.5$, $\tau_0 = 5\Gamma_0^{-1}$ and $T/T_0 = 2.0, 1.1, 1.03$. (b) Gaussian density of states. A plot of $C_{\rm eq}C_\beta(t)$ against $\log_{10}(\Gamma_0 t)$ for $q\xi_0 = 0.5$, $\tau_0 = 5\Gamma_0^{-1}$ and $T/E_{\rm c} = 0.5, 0.25, 0.15$. Note that the plateau observed for the lowest temperature eventually decays to zero.

where the two times only appear now through the combination (t/t_w). This must be contrasted with the situation prevailing for $T > T_0$, where as soon as $t_w \gg \tau(T)$, $P(E, t_w)$ ceases to depend on t_w and the correlation function only depends on the time difference t, and is plotted in figure 1(a) for different values of T/T_0. (We have included a 'fast' relaxation inside each trap—seen as a simple harmonic well, in order to mimic the β regime of glasses.)

For the case of the exponential density of states, using equation (21) we recover the explicit formula derived in [26] through another approach

$$\Pi(t_w + t, t_w) \simeq \frac{\sin \pi x}{\pi} \int_{(t/t_w)/(1+(t/t_w))}^{1} \mathrm{d}v (1-v)^{x-1} v^{-x} \tag{24}$$

with the asymptotic behaviours

$$\Pi(t_w + t, t_w) \simeq 1 - \mathrm{sinc}[\pi(1-x)] \left(\frac{t}{t_w}\right)^{1-x} \qquad \text{for } t \ll t_w \tag{25}$$

$$\Pi(t_w + t, t_w) \simeq \mathrm{sinc}[\pi x] \left(\frac{t}{t_w}\right)^{-x} \qquad \text{for } t \gg t_w \tag{26}$$

where we have introduced the notation $\mathrm{sinc}[u] \equiv \sin u/u$.

2.5. The case of Gaussian density of states at low temperature: stretched exponentials and interrupted aging

Let us consider the interesting case of a Gaussian density of states

$$\rho(E) = \frac{2}{\sqrt{\pi} E_c} \mathrm{e}^{-(E/E_c)^2} \tag{27}$$

which corresponds to a situation where the energy barriers results from a (weak) interaction with many neighbours. Strictly speaking, there is a stationary distribution for any positive temperature (6):

$$P_{\mathrm{eq}}(E) = \omega_{\mathrm{eq}}(\beta) \frac{2}{\sqrt{\pi} E_c} \mathrm{e}^{+\beta E} \mathrm{e}^{-(E/E_c)^2} \equiv \mathcal{N} \mathrm{e}^{-(E-E^*)^2/E_c^2} \qquad \text{where } E^* = \tfrac{1}{2} E_c(\beta E_c). \tag{28}$$

However, at very low temperature $T \ll E_c$, there exists an approximate aging behaviour on the time interval satisfying

$$0 \ll \ln(\Gamma_0 t) \ll (\beta E_c)^2 \tag{29}$$

but this aging phenomenon progressively disappears as time becomes large. The physical reason is the following: as can be seen directly from $\rho(E)$, the distribution of relaxation times $\tau = \Gamma_0^{-1} \mathrm{e}^{\beta E}$ can be written (for large t) as

$$\Psi(\tau) \underset{\tau \to \infty}{\propto} \tau^{-1-\ln(\Gamma_0 \tau)/(\beta E_c)^2}. \tag{30}$$

Hence, the parameter $x = T/T_0$ defined in the previous paragraphs is replaced by a slowly-varying function $x(t) = 2\log(\Gamma_0 t)/(\beta E_c)^2$. (The factor two comes from a more careful analysis—see below.) The above inequality thus corresponds to the case where $x(t) \ll 1$, where aging effects are indeed expected.

More explicitly, one may look for an approximate scaling solution of (1) analogous to (17)

$$P(E, t) \simeq \beta u \phi_t(u) \tag{31}$$

where $u = \mathrm{e}^{\beta E}/\Gamma_0 t$ and where the function ϕ_t is supposed to vary slowly in time. More precisely, in the resulting equation for ϕ_t

$$-ut \frac{\partial \phi_t}{\partial t} + u^2 \frac{\partial \phi_t}{\partial u} + (u-1)\phi_t(u) = -\frac{1}{\beta} \rho\left(\frac{\log(\Gamma_0 t u)}{\beta}\right) \int_1^\infty \frac{\mathrm{d}v}{v} \phi_t\left(\frac{v}{\Gamma_0 t}\right) \tag{32}$$

we assume that the first term containing the time derivative is negligible in comparison to others. We now, therefore, consider the simplified equation

$$u^2\frac{\mathrm{d}\phi_t}{\mathrm{d}u}+(u-1)\phi_t(u)\simeq-\frac{1}{\beta}\rho\left(\frac{\log(\Gamma_0 t u)}{\beta}\right)\int_1^\infty\frac{\mathrm{d}v}{v}\phi_t\left(\frac{v}{\Gamma_0 t}\right) \tag{33}$$

and we will discuss the validity of the hypothesis made on the explicit solution obtained for (33).

For $\Gamma_0 t\gg 1$, the function ϕ_t must present the following asymptotic behaviour,

$$\phi_t(u)\underset{u\to 0}{\simeq}A(t)u^{-x(t)}\,\mathrm{e}^{-(\log u/\beta E_\mathrm{c})^2} \tag{34}$$

and (33) becomes

$$u^2\frac{\mathrm{d}\phi_t}{\mathrm{d}u}+(u-1)\phi_t(u)\simeq A(t)u^{-x(t)}\,\mathrm{e}^{-(\log u/\beta E_\mathrm{c})^2}. \tag{35}$$

The solution reads

$$\phi_t(u)=A(t)\frac{1}{u}\,\mathrm{e}^{-1/u}\int_0^{1/u}\mathrm{d}y\;y^{x(t)-1}\,\mathrm{e}^{-(\log y/\beta E_\mathrm{c})^2}\,\mathrm{e}^{y} \tag{36}$$

where $A(t)$ is determined through the normalization condition $\int_0^\infty \mathrm{d}u\,\phi_t=1$

$$\frac{1}{A(t)}=\int_0^\infty\mathrm{d}y\,\mathrm{e}^{-y}\int_{-\infty}^{+\infty}\mathrm{d}z\,\frac{\mathrm{e}^{zx(t)}}{\mathrm{e}^z+y}\,\mathrm{e}^{-(z/\beta E_\mathrm{c})^2}. \tag{37}$$

The asymptotic behaviours of $\phi_t(u)$ are respectively given by (34) for $u\to 0$, and by the following for $u\to\infty$

$$\phi_t(u)\underset{u\to\infty}{\simeq}\frac{A(t)}{x(t)}u^{-1-x(t)}\,\mathrm{e}^{-(\log u/\beta E_\mathrm{c})^2}. \tag{38}$$

We may now estimate the order of the term dropped in (32):

$$\left|ut\frac{\partial\phi_t}{\partial t}\right|=\left|ut\frac{\mathrm{d}x(t)}{\mathrm{d}t}\frac{\partial\phi_t}{\partial x(t)}\right|=\left|\frac{2}{(\beta E_\mathrm{c})^2}u\frac{\partial\phi_t}{\partial x(t)}\right|\sim\left|\frac{2\log u}{(\beta E_\mathrm{c})^2}u\phi_t(u)\right|. \tag{39}$$

The comparison with other terms of (32) shows that the approximation (33) holds at low temperature $(\beta E_\mathrm{c})^2\gg 1$, and not too large u, i.e. $\log u/(\beta E_\mathrm{c})^2\ll 1$,

Let us now explain how our analysis is compatible with the relaxation towards the equilibrium distribution (28). At low temperature ($\beta E_\mathrm{c}\gg 1$), this equilibrium distribution is concentrated in a region of scale E_c around the central value $E^*=\beta E_\mathrm{c}^2/2$ which is very large compared to E_c. This is why during a very long time interval, the probability distribution $P(E,t)$ is pushed towards deeper and deeper traps as if there were no equilibrium distribution. More precisely, as soon as $\Gamma_0 t\gg 1$, $P(E,t)$ enters an aging-like scaling regime in the variable $u=\mathrm{e}^{\beta E}/\Gamma_0 t$ as in the case of the exponential distribution below the transition ($T<T_0$). From the above remark concerning the shape of $\Psi(\tau)$ (equation (30)), one expects that the aging behaviour disappears as $x(t)$ gets close to the value 1. Then the normalizability of $P_\mathrm{eq}(E)$ becomes noticeable, the probability distribution $P(E,t)$ begins to stabilize, and the scaling variable $u=\mathrm{e}^{\beta E}/\Gamma_0 t$ loses its meaning. In the limit $x(t)\to\infty$, the function ϕ_t becomes concentrated in the region $u\to 0$, and the asymptotic behaviour (34) precisely corresponds to the equilibrium distribution (28) when one replaces u by $\mathrm{e}^{\beta E}/\Gamma_0 t$.

We now turn to the correlation function Π defined in (22). For $\Gamma_0 t_w\gg 1$, one has in terms of the function $\phi_{t_w}(u)$ of equation (36)

$$\Pi(t_w+t,t_w)\simeq\int_0^\infty\mathrm{d}u\,\phi_{t_w}(u)\,\mathrm{e}^{-(1/u)(t/t_w)}. \tag{40}$$

Here, in contrast to the real aging behaviour (23), $\Pi(t_w + t, t_w)$ is not a function of (t/t_w) only, since the function ϕ_{t_w} slowly varies with the waiting time t_w. In particular, using the asymptotic behaviours (34) and (38), one has respectively at short time $t \ll t_w$

$$\Pi(t_w + t, t_w) \underset{t \ll t_w}{\simeq} 1 - A(t_w) \int_0^\infty \mathrm{d}u\, u^{-x(t_w)}\, \mathrm{e}^{-(\log u/\beta E_c)^2} (1 - \mathrm{e}^{(-1/u)(t/t_w)}) \tag{41}$$

and at large time $t \gg t_w$

$$\Pi(t_w + t, t_w) \underset{t \gg t_w}{\simeq} \frac{A(t_w)}{x(t_w)} \int_0^\infty \mathrm{d}y\, y^{x(t_w)-1}\, \mathrm{e}^{-(\log y/\beta E_c)^2}\, \mathrm{e}^{-y(t/t_w)}. \tag{42}$$

To obtain more explicit behaviours, we specialize to the interesting regime $0 < x(t_w) < 1$ where the aging-like behaviour occurs. In this regime, the normalization constant (37) may be expanded according to

$$\frac{1}{A(t_w)} = \left(1 - \frac{1}{(\beta E_c)^2} \frac{\partial^2}{\partial x(t_w)^2} + \cdots\right) \int_0^\infty \mathrm{d}y\, \mathrm{e}^{-y} \int_0^{+\infty} \mathrm{d}z\, \frac{z^{x(t_w)}}{z + y} \tag{43}$$

$$= \exp\left[- \left(\frac{1}{\beta E_c} \frac{\partial}{\partial x(t_w)} \right)^2 \right] \left[\frac{\pi \Gamma(x(t_w))}{\sin \pi x(t_w)} \right]. \tag{44}$$

The explicit expression of the first correction in the small parameter $1/(\beta E_c)^2$ shows that the approximation

$$A(t_w) \simeq \frac{\sin \pi x(t_w)}{\pi \Gamma(x(t_w))} \tag{45}$$

is valid for $1/\beta E_c \ll x(t_w) \ll 1 - 1/\beta E_c$.

The two asymptotic expressions (41) and (42) may be analysed in the same way. At short times

$$\Pi(t_w + t, t_w) \underset{t \ll t_w}{\simeq} 1 - A(t_w) \exp\left[- \left(\frac{1}{\beta E_c} \frac{\partial}{\partial x(t_w)} \right)^2 \right] \left[\frac{\Gamma(x(t_w))}{1 - x(t_w)} \left(\frac{t}{t_w} \right)^{1 - x(t_w)} \right] \tag{46}$$

can be approximated by

$$\Pi(t_w + t, t_w) \underset{t \ll t_w}{\simeq} 1 - \mathrm{sinc}[\pi(1 - x(t_w))] \left(\frac{t}{t_w} \right)^{1 - x(t_w)} \tag{47}$$

for $1/\beta E_c \ll x(t_w) \ll 1 - 1/\beta E_c$ and for t/t_w not exponentially small in $(\beta E_c)^2$.

At large times,

$$\Pi(t_w + t, t_w) \underset{t \gg t_w}{\simeq} \frac{A(t_w)}{x(t_w)} \exp\left[- \left(\frac{1}{\beta E_c} \frac{\partial}{\partial x(t_w)} \right)^2 \right] \left[\Gamma(x(t_w)) \left(\frac{t}{t_w} \right)^{-x(t_w)} \right] \tag{48}$$

may be reduced to

$$\Pi(t_w + t, t_w) \underset{t \gg t_w}{\simeq} \mathrm{sinc}[\pi x(t_w)] \left(\frac{t}{t_w} \right)^{-x(t_w)} \tag{49}$$

for $1/\beta E_c \ll x(t_w) \ll 1 - 1/\beta E_c$ and t/t_w not exponentially large in $(\beta E_c)^2$.

As anticipated, these expressions are very similar to those obtained for the exponential density of states, provided one defines an effective time dependent parameter $x(t)$. (Actually, when $|\ln(t/t_w)| \gg 1$, a slightly more accurate expression of (47) and (49) is obtained by replacing the exponent $x(t_w)$ by $(x(t)/2)$, in agreement with the naive interpretation of equation (30).)

Let us now concentrate on times such that $x(t_w) > 1$: aging effects cease, and the stationary dynamics resume. The resulting correlation function

$$C_{\rm eq}(t) = \int_0^\infty {\rm d}E\, \omega_{\rm eq}(\beta)\, {\rm e}^{\beta E} \frac{2}{\sqrt{\pi}\, E_{\rm c}}\, {\rm e}^{-(E/E_{\rm c})^2} \exp[-(\Gamma_0\, {\rm e}^{-\beta E})t] \tag{50}$$

is plotted in figure 1(b) and displays very interesting features. To a very good approximation, one may replace the effect of the last exponential by a simple cut-off:

$$C_{\rm eq}(t) \simeq \int_{\ln(\Gamma_0 t)/\beta}^\infty {\rm d}E\, \omega_{\rm eq}(\beta)\, {\rm e}^{\beta E} \frac{2}{\sqrt{\pi}\, E_{\rm c}}\, {\rm e}^{-(E/E_{\rm c})^2}. \tag{51}$$

One thus finds the following behaviour:

$$\frac{\partial C_{\rm eq}(t)}{\partial \ln(\Gamma_0 t)} \simeq -\frac{1}{\sqrt{\pi}\beta E_{\rm c}} \exp[-(\ln(t/\tau(T))/\beta E_{\rm c})^2] \tag{52}$$

where $\tau(T) \equiv (1/\Gamma_0)\exp((\beta E_{\rm c})^2/2)$ is the time scale associated with the α regime. As mentioned in the introduction, the experimental data on $\tau(T)$ can indeed be fitted by such a law [4]. Note, however, that the 'terminal' time scale, defined as $\tau_1 = \int_0^\infty C_{\rm eq}\, {\rm d}t$, and extracted, in supercooled liquids, from viscosity measurements, is given by $\tau_1 \simeq (1/\Gamma_0)\, {\rm e}^{\frac{3}{4}(\beta E_{\rm c})^2}\tau(T)$ at low temperature†.

Equation (52) shows that the correlation functions obtained for different temperatures approximately fall onto a master curve if one plots them as a function of $t/\tau(T)$, provided the variation of $\beta E_{\rm c}$ (compared to that of $\tau(T)$) can be neglected. Furthermore, as emphasized in [28], the shape of the relaxation function given by equation (52) is actually very close, in the region where t and $\tau(T)$ are not too different, to a *stretched exponential*‡ $C_{\rm eq}(t) = \exp -(t/\tau(T))^{\alpha(T)}$, provided one identifies $\alpha(T) \simeq (1 + (\beta E_{\rm c})^2/2)^{-\frac{1}{2}}$ [28].

These features are again reminiscent of the experimental ones, where an approximate scaling of $C_{\rm eq}(t)$ for various temperatures can indeed be achieved, and where a stretched exponential (or Kohlrausch) form for the decay in the α regime is quite often proposed (see the discussion in [1]). Interestingly, the exponent α extracted from experimental fits has a tendency to decrease as the temperature is decreased (see, e.g., figure 16 of [29]).

Finally, let us stress that many of the results of the present section are not restricted to the Gaussian form of the density of states, and would still hold if $\rho(E) \propto \exp -(E/E_{\rm c})^{1+\nu}$, $\nu > 0$ with $(\beta E_{\rm c})^2$ replaced by $(\beta E_{\rm c})^{1+1/\nu}$ (up to ν dependent prefactors). Such a generalized model has been recently discussed in the context of glasses in [30]. An interesting point to notice is that the variance σ^2 of the log-normal form of $\partial C_{\rm eq}(t)/\partial \ln t$ is related to its maximum through $\sigma^2 = (1/\nu)\ln\tau(T)$. Such a relation was also obtained by Souletie using different arguments [31], and is not incompatible with experimental data.

The conclusion of this section is that stretched exponential decay and scaling of the relaxation curves can occur in the absence of any criticality or cooperative effects, and arises even when the distribution of local energy barriers decays quite fast for large E. An interesting consequence is the appearance of 'interrupted aging' effects for low temperatures, with a correlation function described by equations (47) and (49). Note that this correlation nearly scales as t/t_w, but with a systematic bias similar to the one discussed in [20].

The above model is, however, oversimplified since:

(i) no *spatial structure* is included; and

† Note, however, that the long time behaviour of $C_{\rm eq}(t)$ is well approximated by $C_{\rm eq}(t) \propto (\tau_2/t)^\mu$ with $\mu = (T/E_{\rm c})^2 \log\Gamma_0 t$ and $\tau_2 = \Gamma_0^{-1}\exp(\beta E_{\rm c})^2$, showing that there is actually quite a bit of freedom to choose the rescaling time factor $\tau(T)$ without affecting too much the quality of the scaling.

‡ Note, the stretched exponential exponent is usually called β, although it describes the α relaxation in glasses, and should furthermore not be confused with the inverse temperature. We have thus proposed in [9] to call it α.

(ii) the particles are independent.

The next two sections aim at discussing, at least partially, these aspects.

3. Model of traps in Euclidean space

3.1. Continuous-time random walks

We now consider that the traps live on a d-dimensional hypercubic lattice. The particle performs a usual random walk on this lattice, but has to wait for a certain trapping time τ before each jump [24]. For a trap of depth E, the distribution of the trapping time τ reads

$$\Psi_E(\tau) = \Gamma_0 \, \mathrm{e}^{-\beta E} \exp[-(\Gamma_0 \, \mathrm{e}^{-\beta E})\tau]. \tag{53}$$

If $\rho(E)$ denotes the probability distribution of the depth E of the traps, the particle performs a random walk among traps with a distribution of trapping times

$$\Psi(\tau) = \int_0^\infty \mathrm{d}E \, \rho(E)\Psi_E(\tau). \tag{54}$$

Note that we consider here a model with annealed disorder, where the energy E for a given site changes at each visit of the particle. This is justified for dimensions $d > 2$, where the particle rarely visits the same site twice. The model that we consider is thus the well known 'continuous time random walk' (CTRW) first introduced by Montroll and Scher, and discussed many times [32]. However, the non-stationary properties of the low-temperature phase (when it exists) has, to our knowledge, never been considered—only the case where the waiting time t_w is zero has been investigated. We shall thus focus on the following natural correlation function,

$$C(\boldsymbol{q}, t + t_w, t_w) = \langle \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}(t+t_w)-\boldsymbol{r}(t_w))} \rangle \tag{55}$$

in particular in the case where there is no equilibrium distribution for (1).

3.2. Correlation function $C(\boldsymbol{q}, t, t_w = 0)$

We shall first recall the result for a vanishing waiting time $t_w = 0$, which is the case usually considered [32]. One has by definition

$$C(\boldsymbol{q}, t, 0) = \langle \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}(t)-\boldsymbol{r}(0))} \rangle = \sum_{\boldsymbol{r}} \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} \mathcal{P}(\boldsymbol{r}, t|\boldsymbol{0}, 0) \tag{56}$$

where the summation is over all sites of the lattice. The probability density may be decomposed according to the number of jumps N as

$$\mathcal{P}(\boldsymbol{r}, t|\boldsymbol{0}, 0) = \sum_{N=0}^{\infty} Q_N(\boldsymbol{r}) R_t(N) \tag{57}$$

where $Q_N(\boldsymbol{r})$ denotes the probability to be at site $\boldsymbol{r}$ after N jumps for a usual random walk on the lattice. $R_t(N)$ denotes the probability to perform N jumps in time t and reads in terms of the distribution $\Psi(\tau)$ of trapping time and Heaviside function θ

$$R_t(N) = \left(\prod_{i=1}^{N+1} \mathrm{d}\tau_i \, \Psi(\tau_i) \right) \theta\left(t - \sum_{i=1}^{N} \tau_i \right) \theta\left(\sum_{i=1}^{N+1} \tau_i - t \right) \tag{58}$$

or more explicitly in terms of the probability density $\Psi_N(\tau)$ of the sum $\tau = \sum_{i=1}^{N} \tau_i$ of N independent trapping times

$$R_t(N) = \int_0^t \mathrm{d}\tau (\Psi_N(\tau) - \Psi_{N+1}(\tau)). \tag{59}$$

The correlation function may then also be decomposed according to the number of jumps N as

$$C(\boldsymbol{q}, t, 0) = \sum_{N=0}^{\infty} R_t(N) \hat{Q}_N(\boldsymbol{q}) \tag{60}$$

where $\hat{Q}_N(\boldsymbol{q})$ denotes the Fourier transform of a usual random walk on a d-dimensional hypercubic lattice of lattice spacing a starting from site $\mathbf{0}$

$$\hat{Q}_N(\boldsymbol{q}) = \sum_{\boldsymbol{r}} \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}(t)} Q_N(\boldsymbol{r}) = \left(\frac{1}{d} \sum_{\mu=1}^{d} \cos(q_\mu a) \right)^N. \tag{61}$$

It is convenient to introduce the Laplace transform

$$\tilde{C}(\boldsymbol{q}, \lambda, 0) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\lambda t} C(\boldsymbol{q}, t, 0) = \sum_{N=0}^{\infty} \hat{Q}_N(\boldsymbol{q}) \tilde{R}_\lambda(N) \tag{62}$$

where

$$\tilde{R}_\lambda(N) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\lambda t} R_t(N) = \frac{1}{\lambda} (\tilde{\Psi}(\lambda))^N [1 - \tilde{\Psi}(\lambda)] \tag{63}$$

in terms of the Laplace transform $\tilde{\Psi}(\lambda)$ of the distribution $\Psi(\tau)$ of trapping time

$$\tilde{\Psi}(\lambda) = \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\lambda \tau} \Psi(\tau). \tag{64}$$

We finally obtain

$$\tilde{C}(\boldsymbol{q}, \lambda, 0) = \frac{1}{\lambda} \frac{1 - \tilde{\Psi}(\lambda)}{1 - \tilde{\Psi}(\lambda) \left((1/d) \sum_{\mu=1}^{d} \cos(q_\mu a) \right)}. \tag{65}$$

3.3. Correlation function $C(\boldsymbol{q}, t + t_w, t_w)$

We now turn to the case of arbitrary waiting time t_w where the correlation function now reads

$$\begin{aligned} C(\boldsymbol{q}, t + t_w, t_w) = {} & \sum_{\boldsymbol{r}} \sum_{\boldsymbol{r}_w} \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r} - \boldsymbol{r}_w)} \\ & \times \int_0^\infty \mathrm{d}E_w \mathcal{P}(\boldsymbol{r}, t + t_w | \boldsymbol{r}_w, E_w, t_w) \mathcal{P}(\boldsymbol{r}_w, E_w, t_w | \mathbf{0}, 0) \end{aligned} \tag{66}$$

where E_w is the energy of the trap needed at t_w. It is convenient to reorganize this expression as follows

$$\begin{aligned} C(\boldsymbol{q}, t + t_w, t_w) = {} & \int_0^\infty \mathrm{d}E_w \sum_{\boldsymbol{r}_w} \mathcal{P}(\boldsymbol{r}_w, E_w, t_w | \mathbf{0}, 0) \\ & \times \sum_{\boldsymbol{r}} \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r} - \boldsymbol{r}_w)} \mathcal{P}(\boldsymbol{r}, t + t_w | \boldsymbol{r}_w, E_w, t_w) \end{aligned} \tag{67}$$

and to decompose the conditional probability $\mathcal{P}(\boldsymbol{r}, t + t_w | \boldsymbol{r}_w, E_w, t_w)$ into

$$\begin{aligned} \mathcal{P}(\boldsymbol{r}, t + t_w | \boldsymbol{r}_w, E_w, t_w) = {} & \delta_{\boldsymbol{r}, \boldsymbol{r}_w} \exp[-(\Gamma_0 \mathrm{e}^{-\beta E_w}) t] \\ & + \int_0^t \mathrm{d}\tau\, \Psi_{E_w}(\tau) \frac{1}{2d} \sum_{\boldsymbol{e}} \mathcal{P}(\boldsymbol{r}, t - \tau | \boldsymbol{r}_w + \boldsymbol{e}, 0). \end{aligned} \tag{68}$$

The first term takes into account the probability to stay in the trap $\boldsymbol{r}_w$ of energy E_w during the time interval $[t_w, t_w + t]$. The second term describes the probability to jump at the time $t_w + t$ out of the trap of energy E_w to go to one of the $2d$ neighbours $\boldsymbol{r}_w + \boldsymbol{e}$ of $\boldsymbol{r}_w$, and where the 'aging' process starts anew. One has, therefore,

$$\sum_{\boldsymbol{r}} \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}-\boldsymbol{r}_w)}\mathcal{P}(\boldsymbol{r}, t + t_w|\boldsymbol{r}_w, E_w, t_w) = \exp[-(\Gamma_0\,\mathrm{e}^{-\beta E_w})t] \tag{69}$$

$$+\int_0^t \mathrm{d}\tau\ \Psi_{E_w}(\tau)\left(\frac{1}{d}\sum_{\mu=1}^{d}\cos(q_\mu a)\right)C(\boldsymbol{q}, t-\tau, t_w = 0). \tag{70}$$

Noticing that

$$\sum_{\boldsymbol{r}_w}\mathcal{P}(\boldsymbol{r}_w, E_w, t_w|\boldsymbol{0}, 0) = P(E_w, t_w) \tag{71}$$

is simply the probability distribution studied in the previous section (1), and that

$$\int_0^\infty \mathrm{d}E_w\ P(E_w, t_w)\exp[-(\Gamma_0\,\mathrm{e}^{-\beta E_w})t] = \Pi(t + t_w, t_w) \tag{72}$$

is the correlation function (22) introduced previously, we obtain finally

$$C(\boldsymbol{q}, t + t_w, t_w) = \Pi(t + t_w, t_w)$$
$$-\left(\frac{1}{d}\sum_{\mu=1}^{d}\cos(q_\mu a)\right)\int_0^t \mathrm{d}\tau\ C(\boldsymbol{q}, t-\tau, t_w = 0)\frac{\partial}{\partial\tau}\Pi(\tau + t_w, t_w). \tag{73}$$

This convolution product leads one to introduce the Laplace transforms

$$\tilde{C}(\boldsymbol{q}, \lambda, t_w) = \int_0^\infty \mathrm{d}t\ \mathrm{e}^{-\lambda t}C(\boldsymbol{q}, t + t_w, t_w) \tag{74}$$

and

$$\tilde{\Pi}(\lambda, t_w) = \int_0^\infty \mathrm{d}t\ \mathrm{e}^{-\lambda t}\Pi(t + t_w, t_w) \tag{75}$$

to get the simpler relation

$$\tilde{C}(\boldsymbol{q}, \lambda, t_w) = \tilde{\Pi}(\lambda, t_w)\left[1 - \lambda\left(\frac{1}{d}\sum_{\mu=1}^{d}\cos(q_\mu a)\right)\tilde{C}(\boldsymbol{q}, \lambda, 0)\right]$$
$$+\left(\frac{1}{d}\sum_{\mu=1}^{d}\cos(q_\mu a)\right)\tilde{C}(\boldsymbol{q}, \lambda, 0). \tag{76}$$

Equation (65) finally gives

$$\tilde{C}(\boldsymbol{q}, \lambda, t_w) = \tilde{\Pi}(\lambda, t_w)\left[\frac{1 - \left((1/d)\sum_{\mu=1}^{d}\cos(q_\mu a)\right)}{1 - \tilde{\Psi}(\lambda)\left((1/d)\sum_{\mu=1}^{d}\cos(q_\mu a)\right)}\right]$$
$$+\frac{1}{\lambda}\frac{1 - \tilde{\Psi}(\lambda)}{\left((1/d)\sum_{\mu=1}^{d}\cos(q_\mu a)\right)^{-1} - \tilde{\Psi}(\lambda)}. \tag{77}$$

Another quantity of interest in glasses is the susceptibility, which is related to the above correlation function through

$$\chi(\boldsymbol{q},\omega,t_w)=1+\mathrm{i}\omega\int_0^\infty \mathrm{d}t\,\mathrm{e}^{\mathrm{i}\omega t}C(\boldsymbol{q},t_w+t,t_w)\equiv 1+\mathrm{i}\omega\tilde{C}(\boldsymbol{q},-\mathrm{i}\omega,t_w) \tag{78}$$

(see, however, [8] for a discussion of the fluctuation-dissipation theorem in this context).

3.4. The case of exponential density of states at low temperature

For the exponential density of states $\rho(E)=\beta_0\exp(-\beta_0 E)$, there is no equilibrium distribution for the process (1) when $x\equiv T/T_0<1$. The correlation function Π is known in this case (24) and its Laplace transform reads

$$\tilde{\Pi}(\lambda,t_w)\simeq\frac{1}{t_w}\int_0^\infty \mathrm{d}u\,\mathrm{e}^{-(\lambda t_w)u}\frac{\sin\pi x}{\pi}\int_{u/(1+u)}^1 \mathrm{d}v(1-v)^{x-1}v^{-x} \tag{79}$$

with the asymptotic behaviours

$$\tilde{\Pi}(\lambda,t_w)\underset{\lambda t_w\ll 1}{\simeq}\frac{1}{\Gamma(1+x)}\frac{(\lambda t_w)^x}{\lambda} \tag{80}$$

$$\tilde{\Pi}(\lambda,t_w)\underset{\lambda t_w\gg 1}{\simeq}\frac{1}{\lambda}\left[1-\frac{1}{\Gamma(x)}\frac{1}{(\lambda t_w)^{1-x}}\right]. \tag{81}$$

The distribution of trapping times

$$\Psi(\tau)=\int_0^\infty \mathrm{d}E\,\rho(E)\Gamma_0\,\mathrm{e}^{-\beta E}\exp[-(\Gamma_0\,\mathrm{e}^{-\beta E})\tau]=\frac{x\Gamma_0}{(\Gamma_0\tau)^{1+x}}\int_0^{\Gamma_0\tau}\mathrm{d}y\,y^x\,\mathrm{e}^{-y} \tag{82}$$

presents the slow algebraic decay

$$\Psi(\tau)\underset{(\Gamma_0\tau)\to\infty}{\simeq}\frac{x\Gamma(1+x)}{\Gamma_0^x\tau^{1+x}} \tag{83}$$

and its Laplace transform presents, therefore, the non-analytic behaviour

$$\tilde{\Psi}(\lambda)=\int_0^\infty \mathrm{d}\tau\,\mathrm{e}^{-\lambda\tau}\Psi(\tau)\underset{(\lambda/\Gamma_0)\to 0^+}{\simeq}1-\frac{1}{\operatorname{sinc}\pi x}\left(\frac{\lambda}{\Gamma_0}\right)^x. \tag{84}$$

Moreover, one has for wavevectors such that $1/q$ is much larger than the lattice spacing a

$$\left(\frac{1}{d}\sum_{\mu=1}^{d}\cos(q_\mu a)\right)\underset{|q|\ll 1/a}{\simeq}1-\frac{a^2}{2d}\boldsymbol{q}^2+\cdots. \tag{85}$$

It is convenient to introduce the time scale t_q

$$(\Gamma_0 t_q)^x=\frac{2d}{(qa)^2} \tag{86}$$

corresponding to the typical time needed by the particle to spread over a region of size $1/q$. There are, therefore, three time scales in (77): λ^{-1}, t_q, t_w. We are interested in the region where all three are much bigger than the microscopic time scale Γ_0^{-1}: $\lambda^{-1}\gg 1/\Gamma_0$, (79)–(84), $t_q\gg 1/\Gamma_0$, (85) and $t_w\gg 1/\Gamma_0$, (79). We still we have to distinguish various

regimes in (77). We give the asymptotic results for the correlation function $C(\boldsymbol{q}, t+t_w, t_w)$ after the inversion of the Laplace transform (77):

$$C(\boldsymbol{q}, t+t_w, t_w) \simeq 1 - \mathrm{sinc}[\pi(1-x)] \left(\frac{t}{t_w}\right)^{1-x} \qquad \text{for } t_q \ll t \ll t_w \quad (87)$$

$$C(\boldsymbol{q}, t+t_w, t_w) \simeq \mathrm{sinc}[\pi x] \left(\frac{t}{t_w}\right)^{-x} \qquad \text{for } t_q \ll t_w \ll t \quad (88)$$

$$C(\boldsymbol{q}, t+t_w, t_w) \simeq 1 - \frac{\mathrm{sinc}[\pi x]}{\Gamma(x)} \frac{t}{t_q^x t_w^{1-x}} \qquad \text{for } t \ll t_w \text{ and } t \ll t_q \quad (89)$$

$$C(\boldsymbol{q}, t+t_w, t_w) \simeq 1 - \frac{\mathrm{sinc}[\pi x]}{\Gamma(1+x)} \left(\frac{t}{t_q}\right)^{x} \qquad \text{for } t_w \ll t \ll t_q \quad (90)$$

$$C(\boldsymbol{q}, t+t_w, t_w) \simeq \Gamma(1+x) \left(\frac{t}{t_q}\right)^{-x} \qquad \text{for } t_w \ll t_q \ll t. \quad (91)$$

There are three interesting points to notice.

• For qa very large, such that $t_q \ll t, t_w$, we find the same asymptotic behaviours as for $\Pi(t+t_w, t_w)$ (25)–(26) in the previous section. Physically, this means that as soon as the particle has jumped once, the rapidly oscillating correlation function averages to zero. Hence, only the particles which have not yet moved contribute to the correlation.

• There are two regimes where the correlation function behaves similarly to a stretched exponential at small times, when $t_q \ll t \ll t_w$ or $t_w \ll t \ll t_q$. The exponent α of this stretched exponential is, however, different in both cases: it is equal to $\alpha = x$ when $t_w \ll t \ll t_q$, and equal to $\alpha = 1 - x$ in the other case.

One should, however, stress once more that the present trap model is unable to explain why there should be a relation between the exponent α describing the decay of the correlation in the α regime which we discuss here, and the shape of the relaxation in the short-time (β) regime, which corresponds to intra-trap dynamics. That such a link exists is one of the major predictions of the MCT, which should also exist deep in the glass [9] phase, where aging effects similar to those discussed here are present.

• The frequency dependent susceptibility defined by equation (78) behaves *in a Cole–Cole fashion*,

$$\chi(\boldsymbol{q}, \omega, t_w = 0) \simeq \frac{1}{1 + (-\mathrm{i}\omega t_q)^x} \quad (92)$$

for $\omega \ll \Gamma_0$, $qa \ll 1$. This has been emphasized in, for example, [33]. However, in the aging regime $\omega t_w \gg 1$, the behaviour of χ is given by

$$\chi(\boldsymbol{q}, \omega, t_w) \simeq \frac{1}{\Gamma(x)(-\mathrm{i}\omega t_w)^{1-x}}. \quad (93)$$

A similar expression was obtained in the context of spin-glasses in [26].

4. Model of interacting particles in traps

4.1. Definition of the model

In the above sections, we have considered models for which the motion of a given particle out of its trap does not affect the potential seen by the others. In order to model this effect, we have proposed in [15] to add to model (1) a diffusion term in energy space proportional

to the mean hopping rate itself, $\omega(t) = \int_0^\infty \mathrm{d}E\, \mathrm{e}^{-\beta E} P(E,t)$. The equation for $P(E,t)$ then reads

$$\frac{1}{\Gamma_0}\frac{\partial P}{\partial t} = -\mathrm{e}^{-\beta E} P + \omega(t)\rho(E) + \omega(t) D \frac{\partial}{\partial E}\left[\rho(E)\frac{\partial P}{\partial E} - P\frac{\mathrm{d}\rho}{\mathrm{d}E}\right]. \tag{94}$$

This diffusion term expresses the fact that every 'hop' induces a small change in all the neighbouring E's. Assuming that the transition rate is proportional to the final density of states, the contribution of such an effect to the master equation *a priori* reads

$$\omega(t)\int \mathrm{d}E'\, \mathcal{T}(|E-E'|)\{P(E',t)\rho(E) - P(E,t)\rho(E')\}.$$

In the limit where the width of $\mathcal{T}(|E-E'|)$ is small, justified in a mean-field limit where the number of neighbours is large, this term reduces to the diffusion-like term in (94), with an effective diffusion constant D proportional to the width of $\mathcal{T}$. More general forms for this diffusion term will be discussed below.

As before, equation (94) has to be supplemented by some initial condition $P(E, t = 0) = P_0(E)$ and by a 'hard wall' boundary condition at $E = 0$,

$$\left[\rho(E)\frac{\partial P(E,t)}{\partial E} - P(E,t)\frac{\mathrm{d}\rho}{\mathrm{d}E}\right]\Bigg|_{E=0} = 0 \tag{95}$$

to ensure the conservation of probability.

4.2. Existence of a stationary distribution

There exists a stationary distribution $P_{\mathrm{eq}}(E)$ at temperature $T = \beta^{-1}$ only if the equation

$$-D\frac{\mathrm{d}^2 P_{\mathrm{eq}}(E)}{\mathrm{d}E^2} + \left[D\frac{\rho''(E)}{\rho(E)} + \frac{\mathrm{e}^{-\beta E}}{\omega_{\mathrm{eq}}\rho(E)}\right] P_{\mathrm{eq}}(E) = 1 \tag{96}$$

supplemented by the boundary condition (95) admits a normalizable solution. The discussion again depends on $T_0 = \beta_0^{-1}$ introduced in (7).

- For $T > T_0$, the asymptotic behaviour at high energy of the solution of (96) reads

$$P_{\mathrm{eq}}(E) \underset{E\to\infty}{\simeq} \omega_{\mathrm{eq}}\, \mathrm{e}^{\beta E}\rho(E) \tag{97}$$

which is normalizable since $\beta < \beta_0$.

- For $T < T_0$, equation (96) can be approximated at high energy by

$$\frac{\mathrm{d}^2 P_{\mathrm{eq}}(E)}{\mathrm{d}E^2} - \frac{\rho''(E)}{\rho(E)} P_{\mathrm{eq}}(E) = -\frac{1}{D}. \tag{98}$$

The general solution reads using arbitrary constants A and B

$$Ag_1(E) + Bg_2(E) + \frac{1}{D}\left[g_1(E)\int_0^E \mathrm{d}v\, g_2(v) + g_2(E)\int_E^\infty \mathrm{d}v\, g_1(v)\right] \tag{99}$$

in terms of the two independent solutions

$$g_1(E) = \rho(E) \qquad \text{and} \qquad g_2(E) = \rho(E)\int_0^E \frac{\mathrm{d}u}{\rho^2(u)} \tag{100}$$

of the homogeneous equation. Taking into account that $\rho(E)$ decays at least exponentially at large E for $\beta_0 < +\infty$, one may show that there is no normalizable solution.

In summary, the conditions for the existence of a stationary distribution in term of T_0 are, therefore, exactly the same as in the case $D = 0$ (7).

The case of an exponential density of states was considered in [15]. For completeness, we give the explicit form of the equilibrium distribution,

$$P_{\rm eq}(E) = \mathcal{N}\left[K_\nu(z)\frac{I_{\nu-1}(z_0)}{K_{\nu-1}(z_0)}\mathcal{K}_\nu(z_0) + K_\nu(z)(\mathcal{I}_\nu(z_0) - \mathcal{I}_\nu(z)) + I_\nu(z)\mathcal{K}_\nu(z)\right] \tag{101}$$

where $\nu = 2T/(T - T_0)$, $z \equiv z_0 \exp(E/\nu T_0)$, and $z_0 = (\nu T_0^{3/2}/\sqrt{D})(\sqrt{\Gamma_0}/\sqrt{\Gamma})$. I_ν and K_ν are the Bessel functions of order ν, and

$$\mathcal{K}_\nu(x) \equiv \int_x^\infty \frac{{\rm d}u}{u} K_\nu(u) \qquad \mathcal{I}_\nu(z) \equiv \int_0^z \frac{{\rm d}u}{u} I_\nu(u).$$

$\mathcal{N}$ and Γ are fixed by the normalization of $P_{\rm eq}(E)$ and the boundary condition (95), which lead to the following equation:

$$\frac{D}{\nu^3 T_0^3} = \frac{I_{\nu-1}(z_0)}{K_{\nu-1}(z_0)}[\mathcal{K}_\nu(z_0)]^2 + 2\int_{z_0}^\infty \frac{{\rm d}u}{u} I_\nu(u)\mathcal{K}_\nu(u). \tag{102}$$

4.3. Aging in the low-temperature phase

When there is no equilibrium distribution (5), we may proceed as in the case $D = 0$ (section 2.4) and look for a scaling solution of the form (17) for equation (94). The resulting equation for the dimensionless function ϕ again only admits some non-trivial limit as $t \to \infty$ if the left-hand side of (18) does. For an example, in the case of an exponential density of states $\rho(E) = \beta_0 \, {\rm e}^{-\beta_0 E}$, in which there is no equilibrium distribution when $x \equiv T/T_0 \in]0, 1[$, the function ϕ must present the singularity (19)

$$\phi(u) \underset{u\to 0}{\simeq} \gamma u^{-x}. \tag{103}$$

The equation for ϕ now generalizes equation (20)

$$\Delta u^3 \frac{{\rm d}^2\phi}{{\rm d}u^2} + [3\Delta u^2 + u^{2+x}]\frac{{\rm d}\phi}{{\rm d}u} + [\Delta(1 - x^3)u + (u - 1)u^x]\phi(u) = -\gamma \tag{104}$$

where $\Delta = \gamma D\beta^3$. The normalization constant γ is determined by the normalization $\int_0^\infty {\rm d}u\, \phi(u) = 1$. In contrast to the case $D = 0$ (21), the solution ϕ cannot be explicitly written. However, its asymptotic behaviour at large u is easily obtained from (104):

$$\phi(u) \underset{u\to\infty}{\simeq} \frac{\gamma}{x} u^{-1-x}. \tag{105}$$

The aging behaviour (23) of the correlation function (22) still holds as a consequence of the scaling form (17). The asymptotic behaviours (103)–(105) of the scaling function ϕ induce the asymptotic expressions generalizing (25) and (26) for the correlation function

$$\Pi(t_w, t_w + t) \simeq 1 - \gamma\frac{\Gamma(x)}{(1 - x)}\left(\frac{t}{t_w}\right)^{1-x} \qquad \text{for } t \ll t_w \tag{106}$$

$$\Pi(t_w, t_w + t) \simeq \gamma\frac{\Gamma(x)}{x}\left(\frac{t}{t_w}\right)^{-x} \qquad \text{for } t \gg t_w. \tag{107}$$

The presence of the diffusion term in (94), therefore, only affects the normalization constant γ, but does not change the asymptotic time dependence of the correlation function $\Pi(t_w, t_w + t)$ of the model (1). This suggests that the difference between the model considered in section (2.4) (corresponding to $D \equiv 0$) and the 'annealed' model considered here is, to some extent, irrelevant.

4.4. Possible generalizations of model (94)

Model (94) may in fact be generalized through the introduction of an energy-dependent diffusion constant, i.e.

$$\frac{1}{\Gamma_0}\frac{\partial P}{\partial t} = -\mathrm{e}^{-\beta E}P + \omega(t)\rho(E) + \omega(t)\frac{\partial}{\partial E}\left(D(E)\left[\rho(E)\frac{\partial P}{\partial E} - P\frac{\mathrm{d}\rho}{\mathrm{d}E}\right]\right) \tag{108}$$

with the same boundary condition (95) as before to ensure the conservation of probability. $\rho(E)$ still corresponds to the equilibrium distribution at infinite temperature ($\beta = 0$). The choice $D(E) = D_0/\rho(E)$, for example, would correspond to an 'entropy' biased diffusion in energy space, with a driving force proportional to $\mathrm{d}\log\rho/\mathrm{d}E$, and an effective diffusion constant independent of E.

A careful study of the existence of a stationary solution to this equation shows that there are two cases.

- If $\lim_{E\to\infty} \log D(E)/E \leqslant 0$, the transition takes place as before at the inverse temperature β_0 (7).
- If $\lim_{E\to\infty} \log D(E)/E > 0$, there exists a stationary distribution at any finite temperature.

In the case where $D(E) = D_0/\rho(E)$ and $\rho(E)$ are exponential, for example, one finds that the large E behaviour of $P_{\mathrm{eq}}(E)$ is of the form:

$$P_{\mathrm{eq}}(E) \underset{E\to\infty}{\simeq} A\,\mathrm{e}^{-\beta_0 E} + B\,\mathrm{e}^{-2\beta_0 E} \tag{109}$$

showing that the correlation function now decays as $C_{\mathrm{eq}}(t) \propto t^{-(1+T/T_0)}$ for large times. Note that the terminal time scale $\tau_1 = \int_0^\infty \mathrm{d}t\, C_{\mathrm{eq}}(t)$ diverges when $T < T_0$, although no asymptotic aging effects appear in this temperature regime.

5. Conclusion

We have studied in this paper a model of particle hopping between energy 'traps' with an arbitrary density of energy barriers $\rho(E)$. As emphasized in [17, 26, 15], the case where $\rho(E)$ decays exponentially is special because it leads to a true dynamical phase transition between a high-temperature phase and a low-temperature aging phase. More generally, however, one expects that for a large class of $\rho(E)$, 'interrupted' aging effects appear at low enough temperatures, with an ergodic time growing faster than exponentially. It would be interesting to look systematically for aging effects experimentally [18]. Furthermore, the relaxation functions have a strongly stretched shape (see, for example, figure 1(b)), which can be fitted as stretched exponentials. The case where the traps are organized on a d-dimensional lattice is slightly more involved, since a new time scale t_q appears, which is a wave-vector-dependent relaxation time. A schematic way of modelling the *interactions* between the particles, reflecting the fact that when one particle moves the potential energy seen by its neighbours changes, was investigated in section 4. The conclusion regarding the existence of a dynamical transition was found to depend on the shape of the effective diffusion constant $D(E)$. In the case where the transition survives, the role of this 'interaction' is irrelevant.

In conclusion, many of the observed features of glassy systems can be accounted for within a picture of independent particles trapped in random potential wells, without any obvious collective effects [4, 5, 12–15]. How deep is the link between this picture and the mode-coupling theory is not yet clear. However, the MCT relies on the existence of a true critical point T_c, with two important consequences which are beyond the grasp of 'trap' models: the strong link between the α and β regime and the critical behaviour of

the plateau value in $C_{\rm eq}(t)$ as $\sqrt{T_{\rm c}-T}$ [1]. These predictions are, however, hard to test since the critical point is supposed to be blurred by 'activated processes'. As suggested in [9], a crucial test of MCT could be performed by working deep below $T_{\rm c}$, where 'activated processes' are frozen but where strong aging effects should appear.

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Appendix. Relaxation towards the equilibrium distribution in model (94)

In the case $D > 0$, equation (94) is nonlinear and cannot be solved exactly. However, we may introduce some approximations to study the relaxation at large time.

A1. Approximate description of the relaxation

To study the relaxation towards the equilibrium distribution $P_{\rm eq}(E)$, we may set

$$P(E,t) = P_{\rm eq}(E) + \eta(E,t) \tag{110}$$

into (94), linearize the corresponding equation in the perturbation η and replace $P_{\rm eq}(E)$ by the expression $P_{\rm eq}(E) \simeq \omega_{\rm eq}(\beta)\,{\rm e}^{+\beta E}\rho(E)$ which happens to be a very good approximation except in the vicinity of $E = 0$. We obtain finally

$$\frac{1}{\Gamma_0}\frac{\partial \eta}{\partial t} = -{\rm e}^{-\beta E}\eta(E,t) + \rho(E)\int_0^\infty {\rm d}u\,{\rm e}^{-\beta u}\eta(u,t) + \omega_{\rm eq}(\beta)D\left[\rho(E)\frac{\partial^2\eta}{\partial^2 E} - \eta\rho''(E)\right]. \tag{111}$$

We may look for the solution through its decomposition onto a relaxation spectrum

$$\eta(E,t) = \int_0^\infty {\rm d}\lambda\; f_\lambda(E)\,{\rm e}^{-\lambda\Gamma_0 t} \tag{112}$$

with the condition

$$\int_0^\infty {\rm d}E\; f_\lambda(E) = 0 \tag{113}$$

to insure the normalization of the probability density $P(E,t)$ (110). The equation for f_λ (111)

$$-D\omega_{\rm eq}(\beta)\frac{{\rm d}^2 f_\lambda}{{\rm d}^2 E} + \left[D\omega_{\rm eq}(\beta)\frac{\rho''(E)}{\rho(E)} + \frac{{\rm e}^{-\beta E}-\lambda}{\rho(E)}\right] f_\lambda(E) = C_\lambda \tag{114}$$

where

$$C_\lambda = \int_0^\infty {\rm d}\lambda f_\lambda(E)\,{\rm e}^{-\beta E} \tag{115}$$

has to be supplemented by the boundary condition (95)

$$\left[\rho(E)\frac{{\rm d}f_\lambda}{{\rm d}E} - f_\lambda\frac{{\rm d}\rho}{{\rm d}E}\right]\Bigg|_{E=0} = 0. \tag{116}$$

In fact, the self-consistency condition (115) is automatically satisfied once f_λ is a solution of (114), (113) and (116).

Equations (114), (113) and (116) imply that

$$(\lambda - \lambda') \int_0^\infty \frac{\mathrm{d}E}{\rho(E)} f_\lambda(E) f_{\lambda'}(E) = 0. \tag{117}$$

The functions $f_\lambda(E)$ including $P_{\mathrm{eq}}(E) = f_0(E)$ are, therefore, orthogonal to one another with respect to the measure $\mathrm{d}E/\rho(E)$. One may choose to orthonormalize this set of functions according to the scalar product

$$\int_0^\infty \frac{\mathrm{d}E}{\rho(E)} f_\lambda(E) f_{\lambda'}(E) = \delta(\lambda - \lambda'). \tag{118}$$

The development of $P(E,t)$ onto this basis then reads

$$P(E,t) = P_{\mathrm{eq}}(E) + \int_0^\infty \mathrm{d}\lambda\, a(\lambda) f_\lambda(E)\, \mathrm{e}^{-\lambda \Gamma_0 t} \tag{119}$$

where the coefficients $a(\lambda)$ are simply obtained through the scalar product with the initial condition $P(E, t = 0)$

$$a(\lambda) = \int_0^\infty \frac{\mathrm{d}E}{\rho(E)} f_\lambda(E) P(E,0). \tag{120}$$

A2. Nature of the relaxation spectrum

To determine the relaxation spectrum, it is convenient to transform (118) into a usual scalar product

$$\delta(\lambda - \lambda') = \int_0^\infty \frac{\mathrm{d}E}{\rho(E)} f_\lambda(E) f_{\lambda'}(E) = \int_{u_0}^\infty \mathrm{d}u\, \Psi_\lambda(u) \Psi_{\lambda'}(u) \tag{121}$$

through the change of variable

$$E \longrightarrow u(E) = \int_{u_0}^E \frac{\mathrm{d}E'}{\sqrt{\rho(E')}} \tag{122}$$

or conversely

$$E(u) = \int_0^u \frac{\mathrm{d}v}{h(v)} \qquad \text{where} \qquad h(u) = \frac{1}{\sqrt{\rho[E(u)]}} \tag{123}$$

and a change of functions

$$f_\lambda(E) \longrightarrow \Psi_\lambda(u) = \frac{f_\lambda[E(u)]}{(\rho[E(u)])^{\frac{1}{4}}}. \tag{124}$$

The equation for the new function $\Psi_\lambda(u)$ (114) is

$$-\frac{\mathrm{d}^2 \Psi_\lambda}{\mathrm{d}^2 u} + \left[V(u) + \frac{\mathrm{e}^{-\beta E(u)} - \lambda}{D \omega_{\mathrm{eq}}(\beta)} \right] \Psi_\lambda(u) = \frac{C_\lambda}{D \omega_{\mathrm{eq}}(\beta)\, [h(u)]^{\frac{3}{2}}} \tag{125}$$

where

$$V(u) = \frac{15}{4} \left(\frac{h'(u)}{h(u)} \right)^2 - \frac{3}{2} \frac{h''(u)}{h(u)} \tag{126}$$

has to be supplemented by the boundary condition at $u_0 = u(E = 0)$

$$\left[\frac{\mathrm{d}\Psi_\lambda}{\mathrm{d}u} + \frac{3}{2} \frac{h'(u)}{h(u)} \Psi_\lambda \right] \Bigg|_{u = u_0} = 0 \tag{127}$$

and by the condition (113)

$$\int_{u_0}^{\infty} \mathrm{d}u \frac{\Psi_\lambda(u)}{[h(u)]^{\frac{3}{2}}} = 0. \tag{128}$$

To discuss the nature of the spectrum, the only important property of the potential $V(u)$ is that

$$V(u) \underset{u\to\infty}{\longrightarrow} 0. \tag{129}$$

This comes from the fact that $\rho(E)$ presents a rapid enough decay as $E \to \infty$ to be integrable.

At infinite temperature ($\beta = 0$), the constant C_λ (115) on the left-hand side of (125) vanishes (113). The asymptotic form of equation (125) as $u \to \infty$,

$$\frac{\mathrm{d}^2\Psi_\lambda}{\mathrm{d}^2 u} = -\left(\frac{\lambda - 1}{D\omega_{\mathrm{eq}}(\beta)}\right)\Psi_\lambda(u) \tag{130}$$

admits two independent oscillatory solutions for $\lambda > 1$, and the boundary condition at u_0 determines the suitable linear combination to be taken for Ψ_λ up to a normalization constant. However, for $\lambda < 1$, only the exponentially damped solution is acceptable at infinity and the boundary condition at u_0 cannot be satisfied. So we finally obtain that the relaxation spectrum at infinite temperature ($\beta = 0$) consists of a continuum above the gap Γ_0

$$P(E,t) = \rho(E) + \int_1^{\infty} \mathrm{d}\lambda\, a(\lambda) f_\lambda(E)\, \mathrm{e}^{-\lambda\Gamma_0 t} \tag{131}$$

to be compared with the previous result for the limiting case $D = 0$ (9).

At finite temperature ($\beta > 0$), the asymptotic form of the homogeneous equation corresponding to (125) in the limit $u \to \infty$

$$\frac{\mathrm{d}^2\chi_\lambda}{\mathrm{d}^2 u} = -\frac{\lambda}{D\omega_{\mathrm{eq}}(\beta)}\chi_\lambda(u) \tag{132}$$

admits always oscillatory solutions for $\lambda > 0$. The relaxation spectrum at finite temperature ($\beta > 0$) consists, therefore, of a continuum starting from $\lambda = 0$ (119). This will produce an algebraic relaxation determined by the behaviour of the function ϕ_λ in the limit $\lambda \to 0$.

A3. Example of the exponential density of states

We may apply the previous general theory to the particular case of the exponential distribution for $\rho(E)$, for which there exists a stationary distribution in the domain of high temperature, $x \equiv \beta_0/\beta > 1$. The change of variables (122)

$$E \longrightarrow u(E) = u_0\, \mathrm{e}^{\beta_0 E/2} \qquad \text{where} \qquad u_0 = \frac{2}{(\beta_0)^{\frac{3}{2}}} \tag{133}$$

and the function $h(u)$ (123)

$$h(u) = \frac{\beta_0}{2}u \tag{134}$$

are simple enough to give an algebraic form to the potential involved in the inhomogeneous Schrödinger equation (125) satisfied by the new function $\Psi_\lambda(u)$:

$$-\frac{\mathrm{d}^2\Psi_\lambda}{\mathrm{d}^2 u} + \left[\frac{15}{4}\frac{1}{u^2} + \frac{1}{D\omega_{\mathrm{eq}}(\beta)}\left(\frac{u_0}{u}\right)^{2/x} - \frac{\lambda}{D\omega_{\mathrm{eq}}(\beta)}\right]\Psi_\lambda(u) = \frac{C_\lambda}{D\omega_{\mathrm{eq}}(\beta)}\left(\frac{2}{\beta_0 u}\right)^{\frac{3}{2}}. \tag{135}$$

At infinite temperature ($\omega_{\rm eq}(\beta) = 1$; $x = \infty$), the function $f_\lambda(E)$ involved in the decomposition (131) reads in terms of Bessel functions J_ν and Y_ν

$$f_\lambda(E) = \mathcal{N}_k \, [Y_1(ku_0)J_2[ku(E)] - J_1(ku_0)Y_2[ku(E)]] \tag{136}$$

with $k = \sqrt{(\lambda - 1)/D}$ and

$$\mathcal{N}_k = \frac{1}{\sqrt{D\beta_0\Gamma_0}} \frac{1}{\sqrt{J_1^2(ku_0) + Y_1^2(ku_0)}} \tag{137}$$

and (131) may be written with the notation $\lambda_k = 1 + Dk^2$ as

$$P(E,t) = \rho(E) + 2D\, \mathrm{e}^{-\Gamma_0 t} \int_0^\infty k \, \mathrm{d}k \, a(\lambda_k) f_{\lambda_k}(E) \, \mathrm{e}^{-D\Gamma_0 k^2 t}. \tag{138}$$

In the limit $k \to 0$, the behaviours $f_{\lambda_k} \propto k^2$ and $a(\lambda_k) \propto k^2$ for a generic initial condition (see (120)) gives the asymptotic algebraic correction to the exponential $\mathrm{e}^{-\Gamma_0 t}$

$$P(E,t) - \rho(E) \underset{t\to\infty}{\propto} \frac{\mathrm{e}^{-\Gamma_0 t}}{(\Gamma_0 t)^3} \tag{139}$$

to be compared with (9). The presence of the diffusion term in (94) representing the interaction between particles thus tends to accelerate the relaxation in comparison to the case (1) of independent particles.

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